International Science and Technology Journal
المجلة الدولية للعلوم والتقنية

Volume 38 العدد
Part 1 المجلد

http://www.doi.org/10.62341/mapd1875

Received	2025/12/29	تم استلام الورقة العلمية في
Accepted	2026/01/19	تم قبول الورقة العلمية في
Published	2026/01/20	تم نشر الورقة العلمية في

Parameters Estimation of Exponential Log-Compound Rayleigh Distribution

Marwah Ahmed Aefa[1*] and Ali Khair S. Alshabani[2]

[1]Department of Mathematics, Faculty of Education, Wadi Alshatti University, Brak, Libya
[2]Department of Mathematics Sciences, School of Basic Sciences, Libyan Academy for Postgraduate Studies, Tripoli, Libya
Correspondence Email Addresses: m.aefa @wau.edu.ly

Abstract:
In this paper, a new continuous probability distribution has been proposed. The proposed distribution is considered as one of the extended distributions of Rayleigh distribution, which is well known in probability Theory and its applications. The distribution under consideration is titled "Exponential Log-compound Rayleigh distribution". proposed. The statistical properties of distribution parameters estimating using Maximum likelihood technique and Fisher information matrix are investigated theoretically. The evaluating of distribution performance is demonstrated on both simulated data and real data. Based on the values of bias, MSE and Fisher information matrix of the parameters estimates, the proposed distribution is considered to be working well.

Keywords and statements:
Exponential, Log-compound, Rayleigh distribution, Maximum likelihood estimation, Performance.

http://www.doi.org/10.62341/mapd1875

تقدير معالم توزيع رايلي الأسي اللوغاريتمي المركب

مروة أحمد عيفة[1]* وعلي خير صابر الشيباني[2]

1- قسم الرياضيات، كلية التربية، جامعة وادي الشاطئ، براك، ليبيا
2- قسم علوم الرياضيات، مدرسة العلوم الأساسية، الاكاديمية الليبية للدراسات العليا، طرابلس، ليبيا
m.aefa@wau.edu.ly

الملخص:
في هذه الورقة البحثية، تم اقتراح توزيع احتمالي متصل جديد. يُعتبر هذا التوزيع أحد التوزيعات المُوسعة لتوزيع رايلي، المعروف في نظرية الاحتمالات وتطبيقاتها. أُطلق على التوزيع قيد الدراسة الحالية "توزيع رايلي الأسي اللوغاريتمي المركب". تمت مناقشة الخصائص الإحصائية لتقدير معالم التوزيع قيد الدراسة بتبني طريقة الامكان الأعظم (الارجحية العظمي) ومصفوفة معلومات نظريًا فيشر. تم تقييم أداء نموذج التوزيع المقترح من خلال تطبيقيه على البيانات مولدة بالمحاكاة وبيانات الحقيقية. بناءً على قيم التحيز، ومتوسط مربع الخطأ، ومصفوفة معلومات فيشر لتقديرات المعلمات، يمكن ان يُعتبر نموذج التوزيع المقترح فعالًا.

الكلمات والجُمل المفتاحية: اسي، المركب–اللوغاريتمي، توزيع رايلي، تقدير الإمكان الأعظم، أداء.

1. Introduction

Statistical distributions are important for parametric inferences and applications to fit real world phenomena. A Hough, several methods for generating new families of distributions have been studied in recent years; these methods were introduced by Alzaatreh *et al*,[3]. However, statistician, often define some new generated families of probability distribution by incorporate one or more extra shape parameters to the baseline model to yield new flexible distributions in order to provide great flexibility in modelling data so as to fit the

http://www.doi.org/10.62341/mapd1875

real-life data. Many authors have proposed and generalized various standard distributions based on the T−X families for example, see [Alzaatreh *et al*. [4], El-Bassiouny *et al.* [5], Ahmad *et al*. [2] and Fatima and Ahmad [6].

Rasheed [7] introduced log compound Rayleigh distribution by logarithmic transformation to the random variable of compound Rayleigh distribution with its basic reliability properties, order statistics and maximum likelihood estimation. The cumulative distribution function (cdf) and the probability density function(pdf) of Log compound Raleigh distribution are given, respectively,

$$G(x;\theta,\lambda) = 1-\lambda^{\theta}(\lambda+e^{2x})^{-\theta} \quad (1)$$

$$g(x;\theta,\lambda) = 2\theta\lambda^{\theta}e^{2x}(\lambda+e^{2x})^{-(\theta+1)}, \quad 0<x<\infty \quad (2)$$

The present article is summarized as follows. In Section 2, an Exponential Log compound Rayleigh distribution is introduced. The maximum likelihood estimation is performed in Section 3. Simulation study is carried out in Section 4. In Section 5, application to real data set is given. Finally, conclusions of the paper are provided in Section 6.

2. Exponential Log Compound Rayleigh Distribution

The cdf of the proposed distribution i.e. exponential log compound Raleigh (ELCR) distribution can be obtained by using the following formula:

$$F(x) = \int_{0}^{\frac{G(x)}{1-G(x)}} \gamma e^{-\gamma t}dt = 1-e^{-\gamma\left[\frac{G(x)}{1-G(x)}\right]}, \quad (3)$$

where $G(x)$ is a baseline cdf, the corresponding family pdf becomes

$$f(x) = \frac{\gamma\, g(x)}{\left(1-G(x)\right)^{2}}e^{-\gamma\left[\frac{G(x)}{1-G(x)}\right]} \quad (4)$$

http://www.doi.org/10.62341/mapd1875

Substituting Eq. (1) into Eq. (3), and after mathematical simplifications we have cdf of the ELCR distribution is given in Eq. (5).

$$F(x) = 1 - e^{-\gamma\left[\frac{(\lambda+e^{2x})^{\theta}}{\lambda^{\theta}}-1\right]}; \quad \gamma, \theta, \lambda > 0 \tag{5}$$

and its pdf counterpart as in Eq. (6).

$$f(x;\gamma,\theta,\lambda) = 2\gamma\theta\lambda^{-\theta}e^{2x}(\lambda+e^{2x})^{\theta-1}e^{-\gamma\left[\frac{(\lambda+e^{2x})^{\theta}}{\lambda^{\theta}}-1\right]}; \quad \gamma, \theta, \lambda > 0 \tag{6}$$

3. Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates of the parameters of the proposed ELCR distribution. For a random sample $X_1, X_2, \dots, X_n$ from a ELCR distribution, the likelihood function is given by

$$\begin{aligned} L(\gamma,\theta,\lambda) &= \prod_{i=1}^{n} f(x;\gamma,\theta,\lambda) \\ &= \left(2\gamma\theta\lambda^{-\theta}\right)^{n} e^{2\sum_{i=1}^{n} x_i} \\ &\quad \times \prod_{i=1}^{n}(\lambda+e^{2x_i})^{\theta-1} \times e^{-\gamma\left[\frac{(\lambda+e^{2x_i})^{\theta}}{\lambda^{\theta}}-1\right]} \end{aligned} \tag{7}$$

Note that in Eq.(7) it is common to write L instead of $L(\gamma,\theta,\lambda)$.

In mathematical statistics it is common to work on natural logarithm (say, ℓ) of likelihood function instead to likelihood function itself when we are solving the following three equations:

$$\frac{\partial \mathrm{L}(\gamma,\theta,\lambda)}{d\gamma} = \frac{\partial \mathrm{L}(\gamma,\theta,\lambda)}{d\theta} = \frac{\partial \mathrm{L}(\gamma,\theta,\lambda)}{d\lambda} = 0.$$

The log likelihood function i.e. natural logarithm (ℓ) of Eq. (7) is shown in Eq. (8).

http://www.doi.org/10.62341/mapd1875

$$\ell = \ln(L) = n\log(2) + n\log(\gamma) + n\log(\theta) - n\theta\log(\lambda) + 2\sum_{i=1}^{n} x_i + (\theta - 1)\sum_{i=1}^{n}\log(\lambda + e^{2x_i}) - \gamma\sum_{i=1}^{n}\left(\frac{(\lambda + e^{2x_i})^{\theta}}{\lambda^{\theta}} - 1\right) \quad (8)$$

The derivatives of Eq. (8) with respect to parameters γ, θ and λ, respectively are given by

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n}\left(\frac{(\lambda + e^{2x_i})^{\theta}}{\lambda^{\theta}} - 1\right)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - n\log(\lambda) + \sum_{i=1}^{n}\log(\lambda + e^{2x_i}) - \gamma\sum_{i=1}^{n}\lambda^{-\theta}(\lambda + e^{2x_i})^{\theta}\{\log(\lambda + e^{2x_i}) - \log(\lambda)\}$$

$$\frac{\partial \ell}{\partial \lambda} = -\frac{n\theta}{\lambda} + (\theta - 1)\sum_{i=1}^{n}\frac{1}{\lambda + e^{2x_i}} - \gamma\sum_{i=1}^{n}\theta\lambda^{-\theta}(\lambda + e^{2x_i})^{\theta}\{(\lambda + e^{2x_i})^{-1} - \lambda^{-1}\}$$

The maximum likelihood estimators for γ, θ and λ can be obtained by equating the three partial derivatives above with zero and solving each of them using a numerical technique.

3.1 Fisher Information Matrix

http://www.doi.org/10.62341/mapd1875

In this section, we will find the Fisher information matrix for obtained confidence interval for the parameters. The Fisher information matrix can be obtained using Equation (8).

Thus, we have

$$I(\hat{\gamma}, \hat{\theta}, \hat{\lambda}) = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \gamma^2} & -\frac{\partial^2 \ell}{\partial \gamma \partial \theta} & -\frac{\partial^2 \ell}{\partial \gamma \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \theta \partial \gamma} & -\frac{\partial^2 \ell}{\partial \theta^2} & -\frac{\partial^2 \ell}{\partial \theta \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \lambda \partial \gamma} & -\frac{\partial^2 \ell}{\partial \lambda \partial \theta} & -\frac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix}_{(\hat{\gamma}, \hat{\theta}, \hat{\lambda})}$$

where the elements of the observed Fisher information matrix are, as follows:

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\frac{n}{\gamma^2}$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2} - \gamma \sum_{i=1}^{n} \lambda^{-\theta} (\lambda + e^{2x_i})^{\theta} \{\log(\lambda + e^{2x_i}) - \log(\lambda)\}^2$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{n\theta}{\lambda^2} - (\theta - 1) \sum_{i=1}^{n} \frac{1}{(\lambda + e^{2x_i})^2} - \gamma \sum_{i=1}^{n} \theta \lambda^{-\theta} (\lambda + e^{2x_i})^{\theta} \{(\theta + 1)\lambda^{-2} + (\theta - 1)(\lambda + e^{2x_i})^{-2} - 2\theta \lambda^{-1} (\lambda + e^{2x_i})^{-1}\}$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \theta} = -\sum_{i=1}^{n} \lambda^{-\theta} (\lambda + e^{2x_i})^{\theta} \{\log(\lambda + e^{2x_i}) - \log(\lambda)\}$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \lambda} = -\sum_{i=1}^{n} \theta \lambda^{-\theta} (\lambda + e^{2x_i})^{\theta} \{(\lambda + e^{2x_i})^{-1} - \lambda^{-1}\}$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \lambda} = -\frac{n}{\lambda} + \sum_{i=1}^{n} \frac{1}{\lambda + e^{2x_i}} -$$

http://www.doi.org/10.62341/mapd1875

$$\gamma\sum_{i=1}^{n}\lambda^{-\theta}(\lambda+e^{2x_i})^{\theta}\left\{\left((\lambda+e^{2x_i})^{-1}-\lambda^{-1}\right)\left\{1\right.\right.$$
$$\left.\left.+\,\theta\left(\log(\lambda+e^{2x_i})-\log(\lambda)\right)\right\}\right\}$$

The Fisher information matrix can be inverted to obtain the estimate of the asymptotic variance-covariance matrix of the maximum likelihood estimates and diagonal elements of $I^{-1}(\hat{\gamma}, \hat{\theta}, \hat{\lambda})$ which provides asymptotic variance of γ, θ and λ respectively. Therefore, this matrix usually used to quantify the amount information of the model that can be obtained from the dataset.

3.2 Approximated Confidence Interval about the Parameters

The asymptotic $100(1-\alpha)\%$ confidence intervals for γ, θ and λ based on their ML estimator method can be constructed using the general formula for large sample property. The three confidence intervals of γ, θ and λ respectively are given below.

$$\hat{\gamma} \pm Z\alpha_{/2}\sqrt{\hat{V}_{\gamma\gamma}}\,,\qquad \hat{\theta} \pm Z\alpha_{/2}\sqrt{\hat{V}_{\theta\theta}}\quad \text{and}\qquad \hat{\lambda} \pm Z\alpha_{/2}\sqrt{\hat{V}_{\lambda\lambda}}$$

where $Z\alpha_{/2}$ is the upper $100\gamma th$ percentile of the standard normal distribution and $\sqrt{\hat{V}_{..}}$ is the standard error of the parameter estimator.

4. Simulation Study

Monte Carlo simulation method has been used, was carried out to illustrate the behavior of maximum likelihood estimation while randomly generating different sample size (20, 40 and 80) from ELCR distribution using inverse cdf method. The process was repeated 1000 times various combinations of parameters are chosen as (0.05, 1.5, 2) and (0.1, 1, 1) for calculation the mean, bias and mean square error (MSE) of estimates. The obtained estimates are shown in Tables 1 and 2. All results are obtained using Mathematic 11 software which is one of the powerful computational software programs widely used in mathematical computations.

http://www.doi.org/10.62341/mapd1875

Table 1: Mean estimates, Bias and MSE of the parameter values ($\gamma = 0.05, \theta = 1.5, \lambda = 2$)

n	Parameter	Mean	Bias	MSE
20	γ	0.052787	0.002787	0.000097
	θ	1.51983	0.01983	0.007903
	λ	2.16186	0.16186	0.179306
40	γ	0.052880	0.002880	0.000093
	θ	1.52522	0.02522	0.005126
	λ	2.17241	0.17241	0.174863
80	γ	0.052742	0.002742	0.000086
	θ	1.53063	0.03063	0.004826
	λ	2.14965	0.14965	0.161432

Table 2: Mean estimates, Bias and MSE of the parameter values ($\gamma = 0.1, \theta = 1, \lambda = 1$)

n	Parameter	Mean	Bias	MSE
20	γ	0.103005	0.003005	0.000625
	θ	1.05559	0.05559	0.059544
	λ	1.18035	0.18035	0.127661
40	γ	0.104529	0.004529	0.000502
	θ	1.06315	0.06315	0.053924
	λ	1.18849	0.18849	0.114195
80	γ	0.104149	0.004149	0.000512
	θ	1.07711	0.07711	0.049006
	λ	1.17632	0.17632	0.112891

From Table 1 and Table 2 we can observe that as sample size increases the values of both Bais and MSE are decreases.

5. Real Data

In this section, we present the application of ELCR distribution to real-world dataset to illustrate the maximum likelihood estimators of unknown parameters. The data have been obtained from Aarset [1]. The dataset represents the lifetimes of 50 devices. The data observations are shown in Table 3.

http://www.doi.org/10.62341/mapd1875

Table 3: The lifetime records of the 50 devices.

0.1,	**0.2,**	**1,**	**1,**	**1,**	**1,**	**1,**	**2,**	**3,**	**6,**
7,	**11,**	**12,**	**18,**	**18,**	**18,**	**18,**	**18,**	**21,**	**32,**
36,	**40,**	**45,**	**46,**	**47,**	**50,**	**55,**	**60,**	**63,**	**63,**
67,	**67,**	**67,**	**67,**	**72,**	**75,**	**79,**	**82,**	**82,**	**83,**
84,	**84,**	**84,**	**85,**	**85,**	**85,**	**85,**	**85,**	**86,**	**86**

The estimated results of the three parameters are presented in the Table (4). The Figure below graphically shows the profile log-likelihood function for the three parameters. The approximate position of each parameter estimate is shown on its relevant plot. In all three plots in the Figure are highly curved which indicate that the likelihood function provides a lot of information about the parameters.

Table 4: Estimates of parameters

Parameter	γ	θ	λ
MLEs	**0.42688**	**0.0105793**	**0.110117**

The estimated variance-covariance matrix is given by

$$I^{-1}(\hat{\gamma}, \hat{\theta}, \hat{\lambda}) = \begin{pmatrix} 0.0626241 & -0.0007263 & 0.0113076 \\ -0.0007263 & 8.97313 \times 10^{-6} & -0.0000980 \\ 0.0113076 & -0.0000980 & 0.0588565 \end{pmatrix}$$

The diagonal of variance-covariance matrix shows the variances of $\hat{\gamma}$, $\hat{\theta}$ and $\hat{\lambda}$ respectively. They are reasonably small values, in particular the $var(\hat{\theta})$. The off-diagonal entries are the bivariate covariances of the three parameters. As we can see that there is a negative relation between $\hat{\theta}$ and the other two parameters $\hat{\gamma}$ and $\hat{\lambda}$ separately.

Based on normality approximation, the constructed 95% confidence interval for the three parameters γ, θ and λ are (-0.06361,0.91737), (0.00471,0.01645) and (-0.36539,0.58562) respectively.

International Science and Technology Journal
المجلة الدولية للعلوم والتقنية
Volume 38 العدد
Part 1 المجلد
http://www.doi.org/10.62341/mapd1875

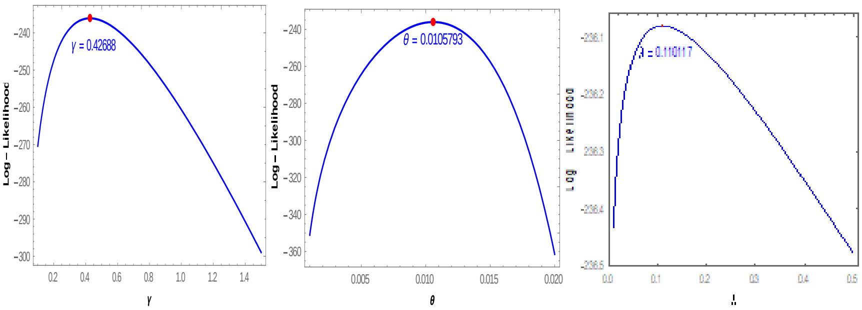

Figure: Profile log-likelihood function of parameters

6. Summary and Conclusion

In the current study a new generalization of Log Compound Rayleigh (LCR) distribution called Exponential Log compound Rayleigh (ELCR) distribution have been introduced. Maximum likelihood estimation method has been used to assess the performance of the estimated parameters and determine the observed information matrix. The simulation study was carried out to assess the performance of the maximum likelihood estimators. The biases and MSEs decrease as the sample size increases and the MLEs of the parameters approached the true parameters. Application of the proposed distribution were illustrated using a real dataset. The performance of the proposed model was demonstrated by a real-world dataset. The values of Fisher information matrix indicate highly curvature of likelihood function which supports that significant amount of information about the proposed model can be obtained from the dataset. In short, based on the obtained results, the adopting of ELCR distribution is expected to improve the modelling of dataset that modellable using Rayleigh distribution and its family.

References

[1**]** Aarset, M.V. (1987). How to Identify a Bathtub Hazard Rate. IEEE Transactions on Reliability.R-36(1).106-108.

http://www.doi.org/10.62341/mapd1875

[2] Ahmad, A., Ahmad, S.P. and Ahmed, A. (2017). Characterization and Estimation of Weibull-Rayleigh Distribution with Applications to Life Time Data. Applied Mathematics& Information Sciences Letters.5(2),71-79.

[3] Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. METRON. 71, 63-79.

[4] Alzaatreh, A., Famoye, F. and Lee, C. (2013). Weibull-Pareto distribution and its applications, Communications in Statistics-Theory and Methods, 42, 1673-1691.

[5] El-Bassiouny, A.H., Abdo, N.F. and H. S. Shahen, H.S. (2015). Exponential Lomax Distribution, International Journal of Computer Applications, 121(13), 24-29.

[6] Fatima, K. and Ahmad, S. P. (2017). Statistical properties of Exponential Rayleigh distribution and its Applications to Medical Science and Engineering. International Conference on Recent Innovations in Science, Agriculture, Engineering and Management.491-506.

[7] Rasheed, N. (2021). Log compound Rayleigh Distribution with Applications to the Reliability Data Sets. Research Journal of Mathematical and Statistical Sciences.9(1).1-8.